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Path integral solutions for deformed Pöschl–Teller-like and conditionally solvable potentials

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Abstract

I discuss in this paper the behaviour of the solutions of the so-called q-hyperbolic potentials, i.e. Pöschl–Teller-like and conditionally solvable potentials, in terms of the path integral formalism. The differences in comparison to the usual Pöschl–Teller-like potentials are investigated, including the discrete energy spectra and the bound-state wavefunctions. We also point out the relation of the q-deformation with curvature on hyperboloids.

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1. Introduction

In this paper I want to discuss some specific generalizations of Pöschl–Teller related potentials. They are based on a q-deformation of the usual hyperbolic potentials and are denoted by (we assume without loss of generality $q > 0$)

$$\sinh_q x = \tfrac{1}{2}(\mathrm{e}^x - q\,\mathrm{e}^{-x}), \qquad \cosh_q x = \tfrac{1}{2}(\mathrm{e}^x + q\,\mathrm{e}^{-x}). \tag{1}$$

Consequently we define

$$\tanh_q x = \frac{\sinh_q x}{\cosh_q x}, \qquad \coth_q x = \frac{\cosh_q x}{\sinh_q x}. \tag{2}$$

Note the relation $\cosh_q^2 x - \sinh_q^2 x = q$, which has the consequence that almost all relations known from the usual hyperbolic functions must be modified. In analogy with the usual hyperbolic functions we have on the one hand

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh_q x = \sinh_q x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sinh_q x = \cosh_q x. \tag{3}$$

However, on the other hand, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh_q x = \frac{q}{\cosh_q^2 x}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\coth_q x = -\frac{q}{\sinh_q^2 x}. \tag{4}$$

These potentials belong to the class of shape-invariant potentials as derived from supersymmetric quantum mechanics [22, 10], and were first introduced by Arai [1], and further discussed by Lévai [27] and Lemieux and Bose [26] in the context of general solutions of the hypergeometric equation. Recently, these potentials have also been discussed by Eğrifes *et al* [7]. The introduction of the parameter q may serve as an additional parameter in describing inter-atomic interactions. The (modified) Pöschl–Teller and so-called conditionally solvable potentials serve as models in molecular and solid-state physics, e.g. [20, 28, 29, 31–33, 35], and are known for a long time. For instance, these potentials model in a simple way diatomic molecules with a finite number of bound state solutions together with scattering states. They are anharmonic, though can be treated analytically and are exactly solvable. Usually all these potentials have two free parameters which can be adjusted to experimental observation. The consideration of the q-deformed potentials allows the incorporation of an additional parameter q. However, as it will turn out, this parameter q often only scales the potential in a simple way, but can be further interpreted as curvature (cf section 3).

We can therefore investigate whether it is possible to introduce the additional parameter q to modify a sample of known potentials which are related to the (modified) Pöschl–Teller potential in order to change the energy-level feature of the potentials. The aim of this paper is to investigate this specific class of shape-invariant potentials, where the path integral method is used as a tool to derive the corresponding Green's function. This is of particular importance in boundary value problems with e.g. Dirichlet boundary conditions (potential V_8).

In the following we present the solution of various potentials. For the first potential we sketch the path integral calculation. The path integral method is of particular convenience: first, because the solutions for the discussed potentials can be stated from the known solutions in the literature, cf [19]; second, the solutions in terms of the Green functions give immediately the energy spectrum and the bound-state wavefunctions; third, also the scattering states are given; however, they will not be stated. For the remaining potentials the explicit path integral evaluation is omitted. We just state the Green function, the bound-state wavefunctions and the energy spectrum. In section 3 we discuss our results; in particular, we point out the observation that the deformation parameter q can be interpreted as curvature on a hyperboloid.

2. Path integral solutions

2.1. The potential V_1

The simplest system of such a deformed potential with bound-state solutions based on the q-deformed hyperbolic functions has the form ($x \in \mathbb{R}$)

$$V_1(x) = -\frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{\cosh_q^2 x} = -\frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{\frac{1}{2}(\mathrm{e}^x + q\,\mathrm{e}^{-x})^2}. \tag{5}$$

Extracting a factor $\sqrt{q}$ we obtain

$$V_1(x) = -\frac{\hbar^2}{2mq}\frac{\lambda^2 - 1/4}{\frac{1}{2}(\mathrm{e}^{-\ln\sqrt{q}+x} + \mathrm{e}^{\ln\sqrt{q}-x})^2}. \tag{6}$$

If we define $y = x - \ln\sqrt{q} \in \mathbb{R}$, we obtain

$$V_1(y) = \frac{V_1(x)|_{q=1}}{q} = -\frac{\hbar^2}{2mq}\frac{\lambda^2 - 1/4}{\cosh^2 y}, \tag{7}$$

and the only effect is a scaling of the potential. The corresponding Lagrangian is changed in the following way:

$$\mathcal{L} = \frac{m}{2}\dot{x}^2 + \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{\cosh_q^2 x} \quad \rightarrow \quad \frac{m}{2}\dot{y}^2 + \frac{\hbar^2}{2mq}\frac{\lambda^2 - 1/4}{\cosh^2 y}. \tag{8}$$

According to [2, 19, 25] the path integral solution is given in terms of the corresponding Green function G of the Feynman kernel. The path integral solution of the potential V_1 is simple, because we can directly apply the path integral solution for the symmetric modified Pöschl–Teller potential [25]. Explicitly we have

$$\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 + \frac{\hbar^2}{2m}\frac{\lambda^2 - \frac{1}{4}}{\cosh_q^2 x}\right)\mathrm{d}t\right] = \int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\mathrm{i}} G^{(V_1)}(x'', x'; E), \tag{9}$$

$$G^{(V_1)}(x'', x'; E) = \frac{m}{\hbar^2}\Gamma\left(\frac{1}{\hbar}\sqrt{-2mE} - \tilde{\lambda} + \frac{1}{2}\right)\Gamma\left(\frac{1}{\hbar}\sqrt{-2mE} + \tilde{\lambda} + \frac{1}{2}\right) \times P_{\tilde{\lambda}-1/2}^{-\sqrt{-2mE}/\hbar}(\tanh y_<) P_{\tilde{\lambda}-1/2}^{-\sqrt{-2mE}/\hbar}(-\tanh y_>), \tag{10}$$

where I have set $\tilde{\lambda}^2 = (\lambda^2 - 1/4)/q + 1/4$. The $P_\nu^\mu(z)$ are Legendre functions. The bound-states are given by

$$\Psi^{(V_1)}(x) = \left(\frac{n - \tilde{\lambda} - \frac{1}{2}}{q}\frac{\Gamma(2\tilde{\lambda} - n)}{n!}\right)^{1/2} P_{\tilde{\lambda}-1/2}^{n-\tilde{\lambda}+\frac{1}{2}}(\tanh_q x), \tag{11}$$

and the energy spectrum is

$$E_n^{(V_1)} = -\frac{\hbar^2}{2m}\left(n - \sqrt{\frac{\lambda^2 - \frac{1}{4}}{q} + \frac{1}{4}} + \frac{1}{2}\right)^2, \tag{12}$$

where $n = 0, 1, \ldots, N_{\max} < \left[\tilde{\lambda} - \frac{1}{2}\right]$ and $[x]$ denotes the integer values of $x \in \mathbb{R}$. We do not state the continuous solutions, cf [19, 25]. We observe that the principal effect consists in a change in the parameter $\lambda \to \tilde{\lambda}$. For $0 < q < 1$ we observe an increase of the number of energy levels, whereas for $q > 1$ there is a decrease of the number of energy levels in comparison to the original $1/\cosh^2 x$ problem.

2.2. The potential V_2

A more complicate version of this potential is the fully modified Pöschl–Teller potential, now in the form of q-deformed hyperbolic functions, i.e. ($x > \ln\sqrt{q}$)

$$V_2(x) = \frac{\hbar^2}{2m}\left(\frac{\lambda^2 - 1/4}{\sinh_q^2 x} - \frac{\nu^2 - 1/4}{\cosh_q^2 x}\right) = \frac{\hbar^2}{2m}\left(\frac{\lambda^2 - 1/4}{\frac{1}{4}(\mathrm{e}^x - q\,\mathrm{e}^{-x})^2} - \frac{\nu^2 - 1/4}{\frac{1}{4}(\mathrm{e}^x + q\,\mathrm{e}^{-x})^2}\right). \tag{13}$$

Performing the transformation $y = x - \ln\sqrt{q} > 0$ yields

$$V_2(y) = \frac{V_2(x)|_{q=1}}{q} = \frac{\hbar^2}{2mq}\left(\frac{\lambda^2 - 1/4}{\sinh^2 y} - \frac{\nu^2 - 1/4}{\cosh^2 y}\right), \tag{14}$$

and the only effect is a scaling of the potential. We express the solution in terms of a path integral, and again the Green function can be stated in a closed form as known from the literature:

$$G^{(V_2)}(x'',x';E)=\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\ \times(q\cosh_q x'\cosh_q x'')^{-(m_1-m_2)}(\tanh_q x'\tanh_q x'')^{m_1+m_2+1/2}\\ \times{}_2F_1\big(-L_\nu+m_1,L_\nu+m_1+1;m_1-m_2+1;q\cosh_q^{-2}x_<\big)\\ \times{}_2F_1\big(-L_\nu+m_1,L_\nu+m_1+1;m_1+m_2+1;\tanh_q^2 x_>\big). \tag{15}$$

I have set $m_{1,2}=\frac{1}{2}(\tilde\lambda\pm\sqrt{-2mE}/\hbar)$, $L_\nu=\frac{1}{2}(\tilde\nu-1)$, $\tilde\lambda^2=(\lambda^2-1/4)/q+1/4$, $\tilde\nu^2=(\nu^2-1/4)/q+1/4$, and ${}_2F_1(a,b;c;z)$ is the hypergeometric function. The bound-states are [13, 25]

$$\Psi_n^{(\eta,\nu)}(x)=N_n^{(\lambda,\nu)}(q^{-1/2}\sinh_q r)^{\lambda+1/2}(q^{-1/2}\cosh_q x)^{n-\nu+1/2}{}_2F_1\big(-n,\tilde\nu-n;1+\tilde\lambda;\tanh_q^2 x\big), \tag{16}$$

$$N_n^{(\lambda,\nu)}=\frac{1}{\Gamma(1+\tilde\lambda)}\left[\frac{2(\tilde\nu-\tilde\lambda-2n-1)\Gamma(n+1+\tilde\lambda)\Gamma(\nu-n)}{\Gamma(\nu-\tilde\lambda-n)n!}\right]^{1/2}, \tag{17}$$

$$E_n=-\frac{\hbar^2}{2m}(2n+\tilde\lambda-\tilde\nu-1)^2,\qquad n=0,1,\ldots,N_{\max}<\left[\frac{1}{2}(\tilde\nu-\tilde\lambda-1)\right]. \tag{18}$$

Note that the coordinate origin is excluded by $x>\ln\sqrt{q}$. Again, we omit the continuous states.

2.3. The potential V_3

For another type of these kinds of potentials (Manning–Rosen potential) which is related to the Coulomb potential in hyperbolic geometry, we define

$$V_3(x)=-\alpha\coth_q x+\frac{\hbar^2}{2m}\frac{\lambda^2-1/4}{\sinh_q^2 x}\\ =-\alpha\frac{e^x+q\,e^{-x}}{e^x-q\,e^{-x}}+\frac{\hbar^2}{2mq}\frac{\lambda^2-1/4}{(e^{-\ln\sqrt{q}+x}-e^{\ln\sqrt{q}-x})^2}. \tag{19}$$

Performing the same transformation as before we get ($x>\ln\sqrt{q}$)

$$V_3(y)=-\alpha\coth y+\frac{\hbar^2}{2mq}\frac{\lambda^2-1/4}{\sinh^2 y}. \tag{20}$$

and now the 'radial' potential strength is modified. We can expect a modification of the spectral properties due to the additional parameter q. The Manning–Rosen potential with deformed hyperbolic functions can be solved by considering a spacetime transformation in the path integral [24]. We have ($x>\ln\sqrt{q}$) for the Green function

$$G^{(V_3)}(x'',x';E)=\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\ \times\left(\frac{2}{\coth_q x'+1}\cdot\frac{2}{\coth_q x''+1}\right)^{(m_1+m_2+1)/2}\\ \times\left(\frac{\coth_q x'-1}{\coth_q x'+1}\cdot\frac{\coth_q x''-1}{\coth_q x''+1}\right)^{(m_1-m_2)/2}$$

$$\times\, {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1-m_2+1; \frac{\coth_q x_> - 1}{\coth_q x_> + 1}\right)$$
$$\times\, {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1+m_2+1; \frac{2}{\coth_q x_< + 1}\right), \qquad (21)$$

where $L_E = -\frac{1}{2} + \sqrt{2m(\alpha - E)}/2$ and $m_{1,2} = \frac{1}{2}\big(2\tilde{\lambda} \pm \frac{1}{\hbar}\sqrt{-2m(\alpha + E)}\big)$, and $\tilde{\lambda}$ is defined as in V_1. The relevant coordinate and time transformations to obtain a path integral formulation in terms of the modified Pöschl–teller potential have the form [11, 25]

$$\frac{1}{2}(1 - \coth y) = -\frac{1}{\sinh^2 r}, \qquad \mathrm{d}t = \tanh^2 r\, \mathrm{d}s. \qquad (22)$$

The wavefunctions and the energy spectrum of the bound states read $(0, 1, \ldots \leqslant N_{\max} < [\sqrt{m\alpha/2}/\hbar - \frac{1}{2}(s+1)]$, $s = 2\tilde{\lambda}$, $m_2 = (1+s)/2$, $m_1 = (1 + (s+2n+1)/2 + 2m\alpha/\hbar^2(s + 2n+1))/2$, note $n + \frac{1}{2} - m_1 < 0)$:

$$\Psi_n(x) = \left[\left(1 + \frac{4m|\alpha|}{\hbar(s+2n+1)^2}\right)\frac{(2m_1 - 2n - s - 2)n!\Gamma(2m_1 - n - 1)}{\Gamma(n+s+1)\Gamma(2m_1 - s - n - 1)}\right]^{1/2}$$
$$\times (1 - q\,\mathrm{e}^{-2x})^{(s+1)/2}\,\mathrm{e}^{-(2x - \ln\sqrt{q})(m_1 - s/2 - n - 1)}\,P_n^{(2m_1 - 2n - s - 2, s)}(1 - 2q\,\mathrm{e}^{-2x}) \qquad (23)$$

and the energy spectrum has the form

$$E_n = -\frac{\hbar^2(s+2n+1)^2}{8m} - \frac{2m\alpha^2}{\hbar^2(s+2n+1)^2}. \qquad (24)$$

The $P_n^{(\alpha,\beta)}$ are Jacobi polynomials. The number of bound states is determined by $N_{\max}$, which depends on α and s. Decreasing s for fixed α is achieved by $0 < q < 1$. Note that the coordinate origin is excluded by $x > \ln\sqrt{q}$.

2.4. The potential V_4

The fourth kind of potential (Rosen–Morse potential) is defined by $(x \in \mathbb{R})$

$$V_4(x) = \beta\tanh_q x - \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{\cosh_q^2 x}$$
$$= \beta\frac{\mathrm{e}^x - q\,\mathrm{e}^{-x}}{\mathrm{e}^x + q\,\mathrm{e}^{-x}} - \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{(\mathrm{e}^{-\ln\sqrt{q}+x} + \mathrm{e}^{\ln\sqrt{q}-x})^2}. \qquad (25)$$

Performing the same transformation as before we get

$$V_4(y) = \beta\tanh y + \frac{\hbar^2}{2mq}\frac{\lambda^2 - 1/4}{\cosh^2 y} \qquad (26)$$

and again only the 'radial' potential strength is modified. For the Rosen–Morse potential in q-deformed hyperbolic functions we obtain $(x \in \mathbb{R})$ for the Green function

$$G^{(V_4)}(x'', x'; E) = \frac{m}{\hbar^2}\frac{\Gamma(m_1 - L_B)\Gamma(L_B + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times \left(\frac{1 - \tanh_q x'}{2}\frac{1 - \tanh_q x''}{2}\right)^{\frac{m_1 - m_2}{2}}\left(\frac{1 + \tanh_q x'}{2}\frac{1 + \tanh_q x''}{2}\right)^{\frac{m_1 + m_2}{2}}$$
$$\times\, {}_2F_1\left(-L_B + m_1, L_B + m_1 + 1; m_1 + m_2 + 1; \frac{1 + \tanh_q x_>}{2}\right)$$
$$\times\, {}_2F_1\left(-L_B + m_1, L_B + m_1 + 1; m_1 - m_2 + 1; \frac{1 - \tanh_q x_<}{2}\right), \qquad (27)$$

where $L_B = -\frac{1}{2} + 2\tilde{\lambda}$ and $m_{1,2} = \sqrt{m/2}(\sqrt{-\beta - E} \pm \sqrt{\beta - E})/\hbar$. The relevant coordinate and time transformations to obtain a path integral formulation in terms of the modified Pöschl–Teller Potential have the form [11, 25] ($r > 0$)

$$\tfrac{1}{2}(1 + \tanh y) = \tanh^2 r, \qquad \mathrm{d}t = \coth^2 r\, \mathrm{d}s. \tag{28}$$

The wavefunctions and the energy spectrum are given by $\big(s \equiv 2\tilde{\lambda};\ 0, \ldots, n \leqslant N_{\max} < \big[\frac{1}{2}(s-1) - \sqrt{m|\beta|/2}/\hbar\big], m_1 = \frac{1}{2}(1+s), m_2 = \frac{1}{2}\big(1 + \frac{1}{2}(s - 2n - 1) - \frac{2mA}{\hbar(s-2n-1)}\big) > \frac{1}{2}\big)$

$$\begin{aligned}\Psi_n = {}& \left[\left(1 - \frac{4m|\beta|}{\hbar(s-2n-1)^2}\right) \frac{(s - 2m_2 - 2n)n!\Gamma(s-n)}{\Gamma(s+1-n-2m_2)\Gamma(2m_2+n)}\right]^{1/2} 2^{n+(1-s)/2} \\ & \times (1 - \tanh_q x)^{\frac{1}{2}s - m_2 - n} (1 + \tanh_q x)^{m_2 - \frac{1}{2}} P_n^{(s-2m_2-2n, 2m_2-1)}(\tanh_q x),\end{aligned} \tag{29}$$

$$E_n = -\left[\frac{\hbar^2(s-2n-1)^2}{8m} + \frac{2m\beta^2}{\hbar^2(s-2n-1)^2}\right]. \tag{30}$$

The number of bound-states is determined by $N_{\max}$, which depends on α and s. Increasing s for fixed β is achieved by $q > 1$.

2.5. The potential V_5

Consequently, the q-deformed hyperbolic Scarf potential [13] is defined by ($x > \ln\sqrt{q}$)

$$V_5(x) = V_0 + V_1 \coth_q^2 x + V_2 \frac{\coth_q x}{\sinh_q x} \quad\to\quad V_0 + V_1 \coth y + \frac{V_2}{\sqrt{q}} \frac{\coth y}{\sinh y}. \tag{31}$$

For the q-deformed hyperbolic Scarf potential we obtain for the Green function ($x > \ln\sqrt{q}$)

$$\begin{aligned} G^{(V_5)}(x'', x'; E) = {}& \frac{2m}{\hbar^2} \frac{\Gamma(m_1 - L_\nu)\Gamma(L_\nu + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)} \\ & \times \left(q^{-1/2} \cosh_q \frac{x'}{2} \cosh_q \frac{x''}{2}\right)^{-(m_1 - m_2)} \left(\tanh_q \frac{x'}{2} \tanh_q \frac{x''}{2}\right)^{m_1 + m_2 + 1/2} \\ & \times {}_2F_1\left(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 - m_2 + 1; q^{1/2} \cosh_q^{-2} \frac{x_<}{2}\right) \\ & \times {}_2F_1\left(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 + m_2 + 1; \tanh_q^2 \frac{x_>}{2}\right) \end{aligned} \tag{32}$$

with $m_{1,2} = \eta/2 \pm \sqrt{V_0 + V_1 - 2mE/\hbar^2}$, where $\eta = \sqrt{V_1 + V_2/\sqrt{q} + 1/4}$, $\nu = \sqrt{V_1 - V_2/\sqrt{q} + 1/4}$, and $L_\nu = \frac{1}{2}(\nu - 1)$. The bound-state wavefunctions and the energy spectrum are given by

$$\begin{aligned}\Psi_n(x) = {}& \left[\frac{(2m_1 - 2m_2 - 2n - 1)n!\Gamma(2m_1 - n - 1)}{2\Gamma(2m_2 + n)\Gamma(2m_1 - 2m_2 - n)}\right]^{1/2} \left(q^{-1/4} \sinh_q \frac{x}{2}\right)^{2m_2 - 1/2} \\ & \times \left(q^{-1/4} \cosh_q \frac{x}{2}\right)^{2n - 2m_1 + 3/2} P_n^{[2m_2 - 1, 2(m_1 - m_2 - n) - 1]}\left(\frac{2q^{1/2}}{\cosh_q^2 \frac{x}{2}} - 1\right),\end{aligned} \tag{33}$$

$$E_n = \frac{\hbar^2}{2m}(V_0 + V_1) - \frac{\hbar^2}{2m}\left[(m_1 - m_2 - n) - \frac{1}{2}\right]^2. \tag{34}$$

Here we denote $n = 0, 1, \ldots, N_{\max} < m_1 - m_2 - 1/2$, $m_1 = \frac{1}{2}(1 + \sqrt{V_1 - V_2/\sqrt{q} + 1/4})$, $m_2 = \frac{1}{2}(1 + \sqrt{V_1 + V_2/\sqrt{q} + 1/4})$ and $\kappa = m_1 - m_2 - n$. In order that bound-states can exist, it is required that $V_2 < 0$. Note that the coordinate origin is excluded by $x > \ln\sqrt{q}$.

2.6. The potential V_6

The q-deformed hyperbolic barrier Potential [31] is defined by ($x \in \mathbb{R}$)

$$\begin{aligned} V_6(x) &= V_0 + V_1 \frac{\tanh_q x}{\cosh_q x} + V_2 \tanh_q^2 x \\ &\to V_0 + \frac{V_1}{\sqrt{q}} \frac{\tanh y}{\cosh y} + V_2 \tanh^2 y. \end{aligned} \tag{35}$$

The q-deformed barrier potential is treated in a similar way. We obtain ($x \in \mathbb{R}$, together with the coordinate transformation $(1 + \mathrm{i} \sinh x)/2 = \cosh^2 r$ in order to obtain a modified Pöschl–Teller potential in the new coordinate $r > 0$ [13])

$$\begin{aligned} G^{(V_6)}(x'', x'; E) = \frac{m}{\hbar^2} &\frac{\Gamma(m_1 - L_\nu)\Gamma(L_\nu + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)} \\ &\times (q^{-1} \cosh_q r' \cosh_q r'')^{-(m_1 - m_2)} (\tanh_q r' \tanh_q r'')^{m_1 + m_2 + \frac{1}{2}} \\ &\times {}_2F_1\big(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 - m_2 + 1; q \cosh_q^{-2} x_<\big) \\ &\times {}_2F_1\big(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 + m_2 + 1; \tanh_q^2 x_>\big) \end{aligned} \tag{36}$$

with $\nu = \sqrt{V_2 + \mathrm{i}V_1/\sqrt{q} + 1/4}, \eta = \nu^*$, $L_\nu = \frac{1}{2}(\nu - 1)$ and $m_{1,2} = \eta/2 \pm \sqrt{V_0 + V_2 - 2mE/\hbar^2}$. Furthermore, we have $\eta/2 \equiv \frac{1}{2}(1 + \lambda)$, with the wavefunctions $\big(\lambda_{R,I} = (\Re, \Im)(\lambda), n = 0, 1, \ldots, N_{\max} < \big[\lambda_R - \frac{1}{2}\big]\big)$

$$\begin{aligned} \Psi_n(x) = &\left[\frac{(2\lambda_R - 2n - 1)n!\Gamma(\lambda - n)}{2\Gamma(2\lambda_R - n)\Gamma(n + 1 - \lambda^*)}\right]^{1/2} \\ &\times \left(\frac{1 + \mathrm{i}q^{-1/2} \sinh_q x}{2}\right)^{\frac{1}{2}(\frac{1}{2} - \lambda)} \left(\frac{1 - \mathrm{i}q^{-1/2} \sinh_q x}{2}\right)^{\frac{1}{2}(\frac{1}{2} - \lambda^*)} \\ &\times P_n^{(-\lambda^*, -\lambda)}(\mathrm{i}q^{-1/2} \sinh_q x) \end{aligned} \tag{37}$$

with the energy spectrum

$$E_n = \frac{\hbar^2}{2m}(V_0 + V_2) - \frac{\hbar^2}{2m} \left\{n + \frac{1}{2} - \sqrt{\frac{1}{2}\left[\sqrt{\left(\frac{1}{4} + V_2\right)^2 + \frac{V_1^2}{q}} + \frac{1}{4} + V_2\right]}\right\}^2. \tag{38}$$

The energy spectrum is modified by the varying q in the V_1^2-term.

2.7. The potential V_7

There are four kinds of conditionally solvable potentials [15, 16] related to the Pöschl–Teller potential type. We introduce ($x \in \mathbb{R}, r > \ln \sqrt{q}, y = x - \ln \sqrt{q}, z = r - \ln \sqrt{q}$) and the first two of them are given by

$$\begin{aligned} V_7(x) &= \frac{\hbar^2}{2m} \left(-\frac{A\, \mathrm{e}^{-x}}{\sqrt{1 + q\, \mathrm{e}^{-2x}}} + \frac{B}{1 + q\, \mathrm{e}^{-2x}} + \frac{C}{(1 + q\, \mathrm{e}^{-2x})^2}\right) \\ &\to \frac{\hbar^2}{2m} \left(-\frac{(A/\sqrt{q})\, \mathrm{e}^{-y/2}}{\sqrt{2 \cosh y}} + \frac{B\, \mathrm{e}^y}{2 \cosh y} + \frac{C\, \mathrm{e}^{2y}}{4 \cosh^2 y}\right), \end{aligned} \tag{39}$$

$$V_7'(x)=\frac{\hbar^2}{2m}\left(-\frac{A}{\sqrt{1+q\,\mathrm{e}^{-2x}}}+\frac{B}{1+q\,\mathrm{e}^{-2x}}+\frac{C}{(1+q\,\mathrm{e}^{-2x})^2}\right)$$
$$\to\frac{\hbar^2}{2m}\left(-\frac{A\,\mathrm{e}^{y/2}}{\sqrt{2\cosh y}}\right)+\frac{B\,\mathrm{e}^y}{2\cosh y}+\frac{C\,\mathrm{e}^{2y}}{4\cosh^2 y}. \tag{40}$$

I have adopted the notation from [15, 16]. The potentials V_7, V_7' may be called 'deformed modified Rosen–Morse potentials II and I', respectively. These potentials are also called 'conditionally solvable' (cf [6, 15, 16, 30] and references therein) because exact solutions can only be found if the parameter C takes the value $C=3/4$. The solution of the path integral for the potential V_7 is related to the solution of the (deformed) hyperbolic Scarf potential [15]. From the Green function of the hyperbolic Scarf-like potential we derive the Green function for the potential V_7,

$$G^{(V_7)}(x'',x';E)=\left(\coth u'\coth u''\right)^{1/2}\frac{2m}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times(\cosh u'\cosh u'')^{-(m_1-m_2)}(\tanh u'\tanh u'')^{m_1+m_2+\frac{1}{2}}$$
$$\times{}_2F_1\left(-L_\nu+m_1,L_\nu+m_1+1;m_1-m_2+1;\frac{1}{\cosh^2 u_<}\right)$$
$$\times{}_2F_1(-L_\nu+m_1,L_\nu+m_1+1;m_1+m_2+1;\tanh^2 u_>) \tag{41}$$

with $\sinh u=\mathrm{e}^y=\mathrm{e}^{x-\ln\sqrt{q}}$, $m_{1,2}=\eta/2\pm\sqrt{V_0+V_1-8mE/\hbar^2}$, $\eta=\sqrt{V_1+V_2+1/4}$, $L_\nu=\frac{1}{2}(\nu-1)$ and $\nu=\sqrt{V_1-V_2+1/4}$, together with the identification $V_0=2mA/\hbar^2-\frac{1}{2}$, $V_1=-\left(2mE/\hbar^2+\frac{1}{4}\right)$, $V_2=-2m\tilde{B}/\hbar^2$. The poles of the Green function determine the energy spectrum, and the corresponding residual give the wavefunctions expansions. The quantization condition is found to read ($\tilde{B}=B/\sqrt{q}$)

$$\sqrt{A-E_n-\frac{3\hbar^2}{8m}}=\frac{1}{2}\left(\sqrt{\tilde{B}-E_n}-\sqrt{-\tilde{B}-E_n}\right)-\frac{\hbar}{\sqrt{2m}}\left(n+\frac{1}{2}\right). \tag{42}$$

This give after some algebra a cubic equation in $(-E_n)$ $\left(\lambda=A+C+\tilde{n}^2, C=-3\hbar^2/8m, \tilde{n}=\hbar\left(n+\frac{1}{2}\right)\big/\sqrt{2m}\right)$

$$4\tilde{n}^2(-E_n)^3+[12\tilde{n}^2(\tilde{n}^2+\lambda)-\lambda^2](-E_n)^2$$
$$+\left[16\tilde{n}^2\lambda(A+C+\lambda)-2(\lambda+4\tilde{n}^2)\left(\lambda^2+\frac{\tilde{B}^2}{4}+4\tilde{n}^2(A+C)\right)\right](-E_n)$$
$$+\left[16\tilde{n}^2\lambda^2(A+C)-\left(\lambda^2+\frac{B^2}{4}+4\tilde{n}^2(A+C)\right)^2\right]=0. \tag{43}$$

We obtain for the energy levels

$$E_n=\sqrt[3]{\sqrt{D}+\frac{Q}{2}}-\sqrt[3]{\sqrt{D}-\frac{Q}{2}}+\frac{R}{3}, \tag{44}$$

$$\left.\begin{aligned}
&D=\left(\frac{P}{3}\right)^3+\left(\frac{Q}{2}\right)^2,\qquad P=\frac{3S-R^2}{3},\qquad Q=\frac{2R^3}{27}-\frac{RS}{3}+T,\\
&R=\frac{12\tilde{n}^2(\tilde{n}^2+\lambda)-\lambda^2}{4\tilde{n}^2},\qquad T=\frac{16\tilde{n}^2\lambda^2(A+C)-[\lambda^2+\tilde{B}^2/4+4\tilde{n}^2(A+C)]^2}{4\tilde{n}^2},\\
&S=\frac{8\tilde{n}^2\lambda(A+C+\lambda)-(\lambda^2+4\tilde{n}^2)(\lambda^2+\tilde{B}^2/4+4\tilde{n}^2(A+C)]}{2\tilde{n}^2}.
\end{aligned}\right\} \tag{45}$$

We omit the details concerning the wavefunctions. bound states exist if $A < 0, 0 < \tilde{B} < |A|$, and the number $N_{\max}$ of bound states is found by requiring $|E_n| > \tilde{B}$.

2.8. The potential V_8

The second set of 'conditionally solvable' potentials is given by

$$V_8(r) = \frac{\hbar^2}{2m}\left(f+1-\frac{f-3/4}{1-q\,\mathrm{e}^{-2r}}+\frac{h_1\,\mathrm{e}^{-r}}{\sqrt{1-q\,\mathrm{e}^{-2r}}}+\frac{C}{(1-q\,\mathrm{e}^{-2r})^2}\right)$$

$$\rightarrow \frac{\hbar^2}{2m}\left(f+1-\frac{(f-3/4)\,\mathrm{e}^{z}}{2\sinh z}+\frac{(h/\sqrt{q})\,\mathrm{e}^{-z/2}}{\sqrt{2\sinh z}}+\frac{C\,\mathrm{e}^{2z}}{4\sinh^2 z}\right), \tag{46}$$

$$V_8'(r) = \frac{\hbar^2}{2m}\left(f+1-\frac{f-3/4}{1-q\,\mathrm{e}^{-2r}}+\frac{h_1}{\sqrt{1-q\,\mathrm{e}^{-2r}}}+\frac{C}{(1-q\,\mathrm{e}^{-2r})^2}\right)$$

$$\rightarrow \frac{\hbar^2}{2m}\left(f+1-\frac{(f-3/4)\,\mathrm{e}^{z}}{2\sinh z}+\frac{h_1\,\mathrm{e}^{z/2}}{\sqrt{2\sinh z}}+\frac{C\,\mathrm{e}^{2z}}{4\sinh^2 z}\right). \tag{47}$$

The potentials V_8, V_8' may be called deformed Manning–Rosen potentials II and I, respectively. The effect of the q-deformation in V_7' and V_8' consists just of a scaling in the coordinates, and thus no new information is obtained. The path integral for the potential V_8 is related to the path integral for the hyperbolic barrier potential as discussed in [16]. The details of its solution are not repeated here again, cf [16]. Due to the fact that its solution is defined in the half-space $\mathbb{R}^+$, we must construct the corresponding Green function in terms of the Green function in the entire $\mathbb{R}$, a method described in [14]. This has also been discussed in detail in [16], which is not repeated here. Hence we obtain $\left(\zeta(z) = \frac{1}{2}(1+\tanh z), z = r - \ln\sqrt{q} > 0\right)$

$$G^{(V_8)}(E)(x'', x'; E) = G(\zeta'', \zeta'; E) - \frac{G(\zeta'', \zeta(0); E)G(\zeta(0), \zeta'; E)}{G(\zeta(0), \zeta(0); E)}, \tag{48}$$

with the Green function $G(E)$ given by

$$\begin{aligned} G(\zeta'', \zeta'; E) &= \frac{m/\hbar^2}{\sqrt{\zeta(z')\zeta(z'')}}\frac{\Gamma(m_1 - L_\nu)\Gamma(L_\nu + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)} \\ &\quad\times\left(\frac{1-\sqrt{\zeta(z')}}{2}\cdot\frac{1-\sqrt{\zeta(z'')}}{2}\right)^{(m_1-m_2)/2} \\ &\quad\times\left(\frac{1+\sqrt{\zeta(z')}}{2}\cdot\frac{1+\sqrt{\zeta(z'')}}{2}\right)^{(m_1+m_2+1/2)/2} \\ &\quad\times {}_2F_1\left(-L_\nu + M_1, L_\nu + m_1 + 1; m_1 + m_2 + 1; \frac{1+\sqrt{\zeta_>(z)}}{2}\right) \\ &\quad\times {}_2F_1\left(-L_\nu + M_1, L_\nu + m_1 + 1; m_1 - m_2 + 1; \frac{1-\sqrt{\zeta_<(z)}}{2}\right). \end{aligned} \tag{49}$$

Here I have used the abbreviations

$$\begin{aligned} L_\nu &= \frac{1}{2}\left(\sqrt{f+1+\mathrm{i}\frac{h_1}{q}-\frac{2m}{\hbar^2}E}-1\right), \\ m_{1,2} &= -\frac{1}{2}\sqrt{f+1-\mathrm{i}\frac{h_1}{q}-\frac{2m}{\hbar^2}E} \pm \sqrt{\frac{1}{4}-f}. \end{aligned} \tag{50}$$

Table 1. Solutions of the path integration of q-deformed potentials.

Potential	Effect of q-deformation	Related problem on hyperboloid
V_1	Scaling	Higgs oscillator
V_2	Deformation of angular momentum number	Higgs oscillator
V_3	Deformation of angular momentum number	Coulomb potential
V_4	Deformation of angular momentum number	Coulomb potential
V_5	Scaling	Superintegrable potential
V_6	Scaling	Superintegrable potential
V_7	Complicated involvment of the parameters	Potential on hyperboloid
V_8	Complicated involvment of the parameters	Potential on hyperboloid

Note that the minus sign in the first term in $m_{1,2}$ is due to the reality condition of the problem [16]. bound-states with energy E_n are determined by the equation

$$ {}_2F_1\big(-L_\nu(E_n)+m_1(E_n), L_\nu(E_n)+m_1(E_n)+1; m_1(E_n)+m_2(E_n)+1; \tfrac{1}{2}\big) = 0. \quad (51) $$

A more detailed numerical investigation of this transcendental equation involving the hypergeometric function is not performed here.

3. Summary and discussion

The results of our investigation of the introduction of the q-deformed hyperbolic potentials show a combination of a shift of the coordinate origin of the potential combined with a scaling of the potential strength. In the cases of the potential V_1 to V_6, with the introduction of the parameter q, the energy levels and the wavefunctions were modified by a nonlinear, albeit simple way. In particular, the energy levels could be easily derived from previous calculations. The cases of the potential V_7 and V_8 were somewhat more difficult, which was due to the fact that the energy levels are determined by a third-order equation and a transcendental equation, respectively. q also entered the expressions nonlinearly. Taking into account the potential V_7' and V_8' we will obtain energy spectra determined by a fourth-order equation and a transcendental equation, modified by a simple shift due to the coordinate translation.

Therefore these potentials can serve as modelling potentials where a finite potential trough is required for particle interaction in molecular, atomic or nuclear physics. This feature is especially seen if the potential is defined in the half-space $x > \ln\sqrt{q}$. Depending on whether $0 < q < 1$ or $q > 1$ the number of energy levels and the ground state energy can be increased or decreased, respectively. We see the convenience of the path integral formalism in the solutions of the deformed potential problems. We can easily use previous results, adapted accordingly to the present problems. In the 'radial' problems the introduction of the parameter q forces the quantum motion to take place in the half-space $x > \ln\sqrt{q}$ and *not* in the half-space $x > 0$. We therefore have introduced an impenetrable finite wall between the particle motion and the coordinate origin, which may be identified for instance with the centre-of-mass location of a molecule. This feature alters the energy spectrum in a nonlinear way; in particular, in the $q = 1$ case there is an integer quantum number $\lambda \equiv l \in \mathbb{N}$. However, this is a phenomenological feature and does not constitute new physics. We summarize the effects on the potentials in table 1. I have also included in the list the relation of the q-deformed potentials to a known potential on the hyperboloid. The Higgs oscillator [20] is the curvature analogue of the usual harmonic oscillator in flat space and the Coulomb potential is the curvature analogue of the usual Coulomb potential in flat space.

One should also keep in mind that the q-deformed hyperbolic potentials can be used to describe curvature in spaces of negative constant curvature, i.e., on hyperboloids (compare also [4] for the interrelation of a deformed algebra and the constant negative curvature in the model of the hyperbolic plane [12]). Let us consider, for instance, the simplest hyperboloid

$$u_0^2 - u_1^2 - u_2^2 = R^2, \qquad u_0 \geqslant 0, \tag{52}$$

which describes one sheet of the double-sheeted hyperboloid $\Lambda^{(2)}$. According to [18] on $\Lambda^{(2)}$ there are nine coordinate systems which allow separation of variables in the Helmholtz, respectively, Schrödinger equation. We consider the usual spherical system ($\tau \in \mathbb{R}, \varphi \in [0, 2\pi)$):

$$\left.\begin{aligned} u_0 &= R\cosh\tau,\\ u_1 &= R\sinh\tau\cos\varphi,\\ u_2 &= R\sinh\tau\sin\varphi, \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} u_0 &= \cosh_q\tau,\\ u_1 &= \sinh_q\tau\cos\varphi,\\ u_2 &= \sinh_q\tau\sin\varphi, \end{aligned}\right. \tag{53}$$

and we observe that with the identification $q = R^2$ the q-deformed spherical coordinate system is a possible separating coordinate system for $\Lambda^{(2)}$. Furthermore, we obtain $-\big(\dot{u}_0^2 - \dot{u}_1^2 - \dot{u}_2^2\big) = q\dot{\tau}^2 + \sinh_q^2\tau\dot{\varphi}^2$. A calculation shows that the introduction of q does not change the energy spectrum features for the free quantum motion on $\Lambda^{(2)}$ (just rescale $m \to m/q$). We can consider the Higgs oscillator $V(\vec{\mathbf{u}}) = (mR^2\omega^2/2)\big(u_1 + u_2^2\big)\big/u_0^2$ (V_1 and V_2 are analogues of the usual harmonic oscillator in a curved space) and the Coulomb potential $V(\vec{\mathbf{u}}) = -(\alpha/R)(u_0\big/\sqrt{u_1^2 + u_2^2}$ [18] (V_3 and V_4, where V_3 is the case for the hyperboloid, and V_4 in imaginary Lobachevsky space [19]), and we find that the identification $R^2 = q$ for the coordinate systems (53) of all spectral properties of the two potentials remains valid. V_5 and V_6 are other superintegrable potentials on the hyperboloid [3, 17, 18, 23]. The potentials V_7 and V_8 do not fall into this special class. Therefore, we observe that the parameter λ corresponds to the angular momentum number l. The effect of the q-deformation shifts the dependence of $l \in \mathbb{Z}$ to some number $\lambda \in \mathbb{R}$. In the other cases the parameters V_i are changed accordingly. However, the effect of taking into account explicitly the curvature R according to (53) has almost the same effect. Therefore we can interpret the deformation parameter q in the hyperbolic potentials as a *curvature* term.

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